

A UNIVERSAL PROPERTY OF THE GROUPS SPIN^c AND MP^c

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ABSTRACT. It is well known that *spinors on oriented Riemannian manifolds cannot be defined as sections of a vector bundle associated with the frame bundle* (see [1]). For this reason spin and spin^c structures are often introduced. In this paper we prove that spin^c structures have a universal property among all other structures that enable the construction of spinor bundles. We proceed to prove a similar result for metaplectic^c structures on symplectic manifolds.

1. INTRODUCTION

Around 1928, while studying the motion of a free particle in special relativity, P.A.M Dirac raised the problem of finding a square root to the three dimensional Laplacian, acting on smooth functions on \mathbb{R}^3 . Finding a square root was necessary in order to study such a system in the quantum mechanics setting. His assumptions were that such a square root ought to be a first order differential operator with constant coefficients.

It is well known that in the Euclidean space \mathbb{R}^n , the problem of finding such a square root involves Clifford algebras and their representations. This square root is often called a *Dirac operator*, and it acts on the representation space for the Clifford algebra (also called *the space of spinors*).

The transition from the flat Euclidean space to a general Riemannian manifold is not obvious. There is no representation of the group $SO(n)$ on the space of spinors which is compatible with the Clifford algebra action. This means that in order to generalize the construction of the Dirac operator to Riemannian manifolds, we must introduce additional structure. More precisely, we need to lift the Riemannian structure (where the group involved in $SO(n)$) to a ‘better’ group G .

It is known that the construction of the Dirac operator can be carried out if our manifold has a spin , a spin^c , or an almost complex structure.

In this paper we answer the question: what is the best (or universal) structure that enables the above process? We show that the group $\text{Spin}^c(n)$ (or a non-compact variant of it) is a universal solution to our problem, in the sense that any other solution will factor uniquely through the spin^c one. This suggests that spin^c structures are the natural ones to consider while quantizing the classical energy observable.

It is important to note that spin^c structures are equivalent to having an irreducible Clifford module on the manifold. This fact appears as Theorem 2.11 in [6]. This is another hint for the universality of the group spin^c .

Interestingly, a similar problem can be stated in the symplectic case, and the universal solution involves Mp^c structures, discussed in [4]. The universality statement and the proof are almost identical to those in the Riemannian case.

This paper is organized as follows. First we introduce the problem of finding a square root for the n -dimensional Laplacian acting on smooth (complex-valued) functions on \mathbb{R}^n . This will motivate the definition of Clifford algebras and the study of their representations. Then we generalize the problem to an arbitrary oriented Riemannian manifold, and explain why more structure is needed in order to define the Dirac operator. Next, we state our main theorem about the universality of the (non-compact variant of the) spin^c group, and deduce a few corollaries. In the last section, we prove a similar result in the symplectic setting. Namely, the universality of the metaplectic^c group.

This work was motivated by the introduction of [1], where the problem of defining a Dirac operator for an arbitrary oriented Riemannian manifold is discussed, and the necessity of additional structure is mentioned. For the study of symplectic Dirac operators, we refer to [3], which is the ‘symplectic analogue’ of [1]. Both [1] and [3] are excellent resources.

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2. THE EUCLIDEAN CASE AND CLIFFORD ALGEBRAS

Consider the negative Laplacian acting on smooth complex valued functions on the n -dimensional Euclidean space

$$\Delta: C^\infty(\mathbb{R}^n; \mathbb{C}) \rightarrow C^\infty(\mathbb{R}^n; \mathbb{C}) \quad , \quad \Delta = - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \quad ,$$

and suppose we are interested in finding a square root for Δ , i.e., we seek an operator

$$P: C^\infty(\mathbb{R}^n; \mathbb{C}) \rightarrow C^\infty(\mathbb{R}^n; \mathbb{C})$$

with $P^2 = \Delta$. Motivated by physics, we assume that P is a first order differential operator with constant coefficients, i.e., that

$$P = \sum_{j=1}^n \gamma_j \frac{\partial}{\partial x_j} \quad , \quad \gamma_j \in \mathbb{C} \quad .$$

A simple computation shows that such a P cannot exist unless $n = 1$, and then $P = \pm i \frac{\partial}{\partial x}$. Indeed, the condition $P^2 = \Delta$ implies that

$$P^2 = \sum_{j,l=1}^n \gamma_j \gamma_l \frac{\partial^2}{\partial x_j \partial x_l} = - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$$

and hence

$$\gamma_j \gamma_l + \gamma_l \gamma_j = 0 \quad , \quad \gamma_j^2 = -1 \quad \text{for all } j \neq l$$

which is impossible, unless $n = 1$ and $\gamma_1 = \pm i$.

One way to modify this problem is to observe that the commutativity of complex numbers ($\gamma_j \gamma_l = \gamma_l \gamma_j$) is the property that made this construction impossible. Therefore, we hope to be able to find such a P if the γ_j ’s are taken to be matrices, instead of complex numbers.

Thus, fix an integer $k > 1$, define the *vector valued Laplacian* as

$$\underline{\Delta}: C^\infty(\mathbb{R}^n; \mathbb{C}^k) \rightarrow C^\infty(\mathbb{R}^n; \mathbb{C}^k) \quad , \quad \underline{\Delta}(f_1, \dots, f_k) = (\Delta f_1, \dots, \Delta f_k) \quad ,$$

and look for a square root

$$\underline{P}: C^\infty(\mathbb{R}^n; \mathbb{C}^k) \rightarrow C^\infty(\mathbb{R}^n; \mathbb{C}^k) \quad .$$

If we assume, as before, that

$$\underline{P} = \sum_{j=1}^n \gamma_j \frac{\partial}{\partial x_j} \quad \text{with} \quad \gamma_j \in M_{k \times k}(\mathbb{C}) \quad ,$$

then $\underline{P}^2 = \underline{\Delta}$ if and only if

$$\gamma_j \gamma_l + \gamma_l \gamma_j = 0 \quad , \quad \gamma_j^2 = -1 \quad \text{for} \quad j \neq l \quad .$$

Those relations are precisely the ones used to define *the Clifford algebra* associated to the vector space \mathbb{R}^n . Here is a more common and general definition of this concept.

Definition 2.1. For a finite dimensional vector space V over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , and a symmetric bilinear map $B: V \times V \rightarrow \mathbb{K}$, define *the Clifford algebra*

$$Cl(V, B) = T(V)/I(V, B)$$

where $T(V)$ is the tensor algebra of V , and $I(V, B)$ is the ideal generated by

$$\{v \otimes v - B(v, v) \cdot 1 : v \in V\} \quad .$$

Remark 2.1.

- (1) If e_1, \dots, e_n is an orthogonal basis for V , then $Cl(V, B)$ is the algebra generated by $\{e_j\}$ with relations

$$e_j e_l + e_l e_j = 0 \quad , \quad e_j^2 = B(e_j, e_j) \quad \text{for} \quad j \neq l \quad .$$

- (2) If $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{R}^n , then denote

$$C_n = Cl(\mathbb{R}^n, -\langle \cdot, \cdot \rangle) \quad \text{and} \quad C_n^c = C_n \otimes \mathbb{C} \quad .$$

Example 2.1. For $n = 3$, let

$$\gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad .$$

Then $\{\gamma_1, \gamma_2, \gamma_3\}$ satisfy the required relations, and $\underline{P} = \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} + \gamma_3 \frac{\partial}{\partial x_3}$ will be a square root of $\underline{\Delta} = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$.

The above discussion suggests that we look for a representation

$$\rho: C_n \rightarrow \text{End}(\mathbb{C}^k) \cong M_{k \times k}(\mathbb{C})$$

of the Clifford algebra C_n . It will be even better if we can find an irreducible one (since every representation of C_n is a direct sum of irreducible ones - see Proposition I.5.4 in [2]). Once we fix such a representation, we can set

$$\gamma_j = \rho(e_j) \quad , \quad \underline{P} = \sum_{j=1}^n \gamma_j \frac{\partial}{\partial x_j}$$

where $\{e_j\}$ is the standard basis for \mathbb{C}^k . The operator \underline{P} , called *a Dirac operator*, will then be a square root of $\underline{\Delta}$.

Here is a known fact about representations of complex Clifford algebras (proofs can be found in [1] and in [2]).

Proposition 2.1. *Any irreducible complex representation of C_n has dimension $2^{\lfloor n/2 \rfloor}$ (where $\lfloor n/2 \rfloor$ is the floor of $n/2$). Up to equivalence, there are two irreducible representations if n is odd, and only one if n is even.*

We conclude that finding a square root for $\underline{\Delta}$ is always possible. It is defined using an irreducible representation of C_n . Note that a choice is to be made if n is odd.

3. MANIFOLDS

3.1. The problem. We would like to generalize our previous construction from the Euclidean space to a smooth n dimensional oriented Riemannian manifold (M, g) . More generally, we look for a complex Hermitian line bundle $S \rightarrow M$ and a smooth map

$$\rho: Cl(TM, -g) \rightarrow End(S)$$

which restricts to an irreducible representation

$$\rho_x: Cl(T_x M, -g_x) \rightarrow End(S_x)$$

on the fibers of S . The notation $Cl(TM, -g)$ stands for *the Clifford bundle* of (M, g) . That is, the vector bundle whose fibers are the Clifford algebra of the tangent space, with respect to the symmetric bilinear map $-g$.

Once such a pair (S, ρ) is found, we can choose a Hermitian connection ∇ on S , and define a Dirac operator acting on smooth sections of S , as follows. Choose a local orthonormal frame $\{e_j\}$ and let

$$\underline{P}: \Gamma(S) \rightarrow \Gamma(S) \quad , \quad \underline{P}(s) = \sum_{j=1}^n \rho(e_j) [\nabla_{e_j} s] \quad .$$

It turns out that \underline{P} is independent of the local frame, and thus gives rise to a globally defined operator on sections of S .

3.2. The search for the vector bundle S . If no additional structure is introduced on our manifold M , then we may try to construct the vector bundle S as an associated bundle to the bundle $SOF(M)$ of oriented orthonormal frames on M . This means that we try to take

$$S = SOF(M) \times_{SO(n)} \mathbb{C}^k$$

where $k = 2^{\lfloor n/2 \rfloor}$ and $SO(n)$ acts on \mathbb{C}^k via a representation

$$\epsilon: SO(n) \rightarrow End(\mathbb{C}^k) \quad .$$

We can use an irreducible representation

$$\rho: C_n \rightarrow End(\mathbb{C}^k)$$

in order to define an action of the Clifford bundle $Cl(TM)$ on S as follows.

For any $x \in M$, $\alpha \in T_x M \subset Cl(T_x M, -g_x)$, $v \in \mathbb{C}^k$ and a frame $f: \mathbb{R}^n \xrightarrow{\sim} T_x M$ in $SOF_x(M)$, let α act on $[f, v] \in S_x$ by

$$(\alpha, [f, v]) \mapsto [f, \rho(f^{-1}(\alpha))v] \quad .$$

This will be a well defined action on S if and only if

$$[f \circ A, \rho((f \circ A)^{-1}\alpha)v] = [f, \rho(f^{-1}(\alpha))(\epsilon(A)v)]$$

for any $A \in SO(n)$. This is equivalent to

$$\epsilon(A) \circ \rho(A^{-1}y) = \rho(y) \circ \epsilon(A)$$

where $y = f^{-1}(\alpha)$, and is an equality between linear endomorphisms of \mathbb{C}^k . To summarize, we look for a representation $\epsilon: SO(n) \rightarrow \text{End}(\mathbb{C}^k)$ with the property that

$$(1) \quad \rho(Ay) = \epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1}$$

for all $A \in SO(n)$ and $y \in \mathbb{R}^n$.

Unfortunately:

Claim 3.1. *For $n \geq 3$, there is no representation $\epsilon: SO(n) \rightarrow \text{End}(\mathbb{C}^k)$ satisfying Equation (1) for all A and y .*

The proof will follow from a more general statement later (see Claim 5.1).

3.3. Introducing additional structure. It seems that in order to construct a vector bundle S over an n dimensional oriented Riemannian manifold (M, g) , on which the Clifford bundle $Cl(TM)$ acts irreducibly, we will have to introduce new structure on our manifold: we will need to lift the structure group from $SO(n)$ to a ‘better’ group G . Here is what we mean by *lifting the structure group*.

Definition 3.1. For an n dimensional oriented Riemannian manifold (M, g) , a *lifting of the structure group to a Lie group G* is a principal G -bundle $\pi: P \rightarrow M$, together with a group homomorphism $p: G \rightarrow SO(n)$ and a smooth map $\pi_1: P \rightarrow \text{SOF}(M)$ such that

$$\pi_1(x \cdot g) = \pi_1(x) \cdot p(g) \quad \text{for} \quad x \in P, g \in G,$$

and such that $\pi = \pi_2 \circ \pi_1$ (where $\pi_2: \text{SOF}(M) \rightarrow M$ is the projection).

In other words, we require that the following diagram will commute, and $\pi = \pi_2 \circ \pi_1$.

$$\begin{array}{ccc} P & \longleftarrow & P \times G \\ \pi_1 \downarrow & & \downarrow \pi_1 \times p \\ \text{SOF}(M) & \longleftarrow & \text{SOF}(M) \times SO(n) \\ \pi_2 \downarrow & & \\ M & & \end{array}$$

Once we have such a lift, we can try to construct our vector bundle of rank $k = 2^{\lfloor n/2 \rfloor}$ as

$$S = P \times_G \mathbb{C}^k,$$

where G acts on \mathbb{C}^k via a representation $\epsilon: G \rightarrow \text{End}(\mathbb{C}^k)$.

This will work if the action of $Cl(TM)$ on S , given by

$$(\alpha, [\tilde{f}, v]) \mapsto [\tilde{f}, \rho(f^{-1}(\alpha))v],$$

is well defined. Here $\alpha \in Cl(T_x M)$, $\tilde{f} \in P_x$, $v \in \mathbb{C}^k$, and $f = p(\tilde{f}): \mathbb{R}^n \rightarrow T_x M$ is a frame in $SO F_x(M)$. As before, ρ is a fixed irreducible complex representation of C_n .

Equation (1), which state the condition ϵ has to satisfy, becomes

$$(2) \quad \rho(p(A)y) = \epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1}$$

for all $y \in \mathbb{R}^n$ and $A \in G$.

To summarize, we look for a Lie group G , and a representation $\epsilon: G \rightarrow \text{End}(\mathbb{C}^k)$ for which Equation (2) is satisfied for all A and y .

4. THE MAIN THEOREM

Given an irreducible representation $\rho: C_n \rightarrow \text{End}(\mathbb{C}^k)$ ($k = 2^{\lfloor n/2 \rfloor}$), we look for a Lie group G , a representation $\epsilon: G \rightarrow \text{End}(\mathbb{C}^k)$, and a homomorphism $p: G \rightarrow SO(n)$ for which Equation (2) is satisfied. Note that this problem is of algebraic flavour and does not involve the manifold, the tangent bundle or the Clifford bundle. Thus, our problem is reduced to one where the unknowns are a Lie group, a representation and a group homomorphism.

As we will see, there is more than one solution to this problem, but only one (up to a certain equivalence) which is universal in the sense that every other solution will factor through the universal one. In this universal solution, the group is

$$G = (Spin(n) \times \mathbb{C}^\times) / K$$

where $Spin(n)$ is the double cover of $SO(n)$ and K is the two element subgroup generated by $(-1, -1)$. This is a noncompact group.

Another way to define this group is as the set of all elements of the form

$$c \cdot v_1 v_2 \cdots v_l \in C_n^c = C_n \otimes \mathbb{C}$$

where $c \in \mathbb{C}^\times$, $l \geq 0$ is even, and each $v_j \in \mathbb{R}^n$ is of (Euclidean) norm 1.

For each element $x \in G$ and $y \in \mathbb{R}^n \subset C_n^c$, we have $Ad_x(y) = x \cdot y \cdot x^{-1} \in \mathbb{R}^n$, and the map

$$y \in \mathbb{R}^n \mapsto Ad_x(y) \in \mathbb{R}^n$$

is in $SO(n)$. This defines a group homomorphism

$$\lambda^c: G \rightarrow SO(n) \quad , \quad x \mapsto Ad_x$$

(see [1] for details).

Finally, note that any $B \in SO(n)$ acts on the Clifford algebra C_n^c in a natural way. This action is induced from the standard action of $SO(n)$ on \mathbb{R}^n . Furthermore, Equation (2) is satisfied for all $y \in \mathbb{R}^n$ and $A \in G$ if and only if it is satisfied for all $y \in C_n^c$ and $A \in G$.

Now we can state the universality property of the group G .

Theorem 4.1. *Fix an irreducible complex representation ρ of C_n^c , and let $k = 2^{\lfloor n/2 \rfloor}$. Then:*

- (1) *For $G = (Spin(n) \times \mathbb{C}^\times) / K$, $p = Ad: G \rightarrow SO(n)$ and $\epsilon = \rho|_G: G \rightarrow \text{End}(\mathbb{C}^k)$, we have*

$$\rho(p(A)y) = \epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1}$$

for all $y \in C_n^c$ and $A \in G$.

- (2) If G' is a Lie group, $p': G' \rightarrow SO(n)$ a group homomorphism, and $\epsilon': G' \rightarrow \text{End}(\mathbb{C}^k)$ a representation, such that

$$\rho(p'(A)y) = \epsilon'(A) \circ \rho(y) \circ \epsilon'(A)^{-1}$$

for all $y \in C_n^c$ and $A \in G'$, then there is a unique homomorphism $f: G' \rightarrow G$ such that

$$p' = p \circ f \quad \text{and} \quad \epsilon' = \epsilon \circ f .$$

Remark 4.1.

- (1) The group G acts on C_n^c via

$$(A, y) \mapsto p(A)y$$

and on $\text{End}(\mathbb{C}^k)$ via

$$(A, \varphi) \mapsto \epsilon(A) \circ \varphi \circ \epsilon(A)^{-1} .$$

Therefore, in part (1) of the theorem we claim that ρ is G -equivariant.

- (2) Part (2) of the theorem implies that the the following two diagrams are commutative.

$$\begin{array}{ccc} & G & \\ \nearrow f & \downarrow p & \\ G' & \xrightarrow{p'} & SO(n) \end{array} \quad \begin{array}{ccc} & G & \\ \nearrow f & \downarrow \epsilon & \\ G' & \xrightarrow{\epsilon'} & \text{End}(\mathbb{C}^k) \end{array}$$

Proof.

- (1) For any $A \in G$ and $y \in C_n^c$ we have

$$\begin{aligned} \rho(p(A)y) &= \rho(\text{Ad}_A(y)) = \rho(A \cdot y \cdot A^{-1}) = \rho(A) \cdot \rho(y) \cdot \rho(A^{-1}) = \\ &= \epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1} \end{aligned}$$

- (2) Fix an element $g \in G'$, and choose an element $A \in \text{Spin}(n)$ for which $p(A) = \text{Ad}_A = p'(g)$. We claim that the endomorphism

$$\rho(A^{-1}) \circ \epsilon'(g): \mathbb{C}^k \rightarrow \mathbb{C}^k$$

is a nonzero (complex) multiple of the identity. To see this, start from the given equality

$$\rho(p'(g)y) = \epsilon'(g) \circ \rho(y) \circ \epsilon'(g)^{-1}$$

which is equivalent to

$$\epsilon'(g) \circ \rho(y) = \rho(\text{Ad}_A(y)) \circ \epsilon'(g)$$

and to

$$[\rho(A^{-1}) \circ \epsilon'(g)] \circ \rho(y) = \rho(y) \circ [\rho(A^{-1}) \circ \epsilon'(g)]$$

for all $y \in C_n^c$.

It is known that any irreducible complex representation of C_n^c must be onto, and hence the last equality means that $\rho(A^{-1}) \circ \epsilon'(g)$ commutes with all endomorphisms of \mathbb{C}^k , and thus must be a multiple of the identity (it is nonzero since it is invertible). Write $\rho(A^{-1}) \circ \epsilon'(g) = c \cdot I$ for $c \in \mathbb{C}^\times$ and define

$$f: G' \rightarrow G \quad , \quad g \mapsto [A, c] \in G .$$

This map is well defined. It is a group homomorphism since if $g_j \in G'$ (for $j = 1, 2$), $A_j \in Spin(n)$ with $p'(g_j) = Ad_{A_j}$ and $c_j \in \mathbb{C}^\times$ satisfy

$$c_j \cdot I = \rho(A_j^{-1}) \circ \epsilon'(g_j)$$

then we have

$$p(g_1 g_2) = Ad_{A_1 A_2}$$

and

$$c_1 c_2 \cdot I = \rho((A_1 A_2)^{-1}) \circ \epsilon'(g_1 g_2) .$$

This implies that $f(g_1 g_2) = f(g_1) f(g_2)$.

Also we have

$$p'(g) = Ad_A = Ad_{c \cdot A} = p(f(g))$$

and

$$\epsilon'(g) = c \cdot \rho(A) = \epsilon(c \cdot A) = \epsilon(f(g))$$

for all $g \in G'$ as needed.

It is not hard to see that our construction implies also the uniqueness of the map f . After all, if such an f exists, and for $g \in G'$ we write $f(g) = [A, c] \in G$, then $p(A) = Ad_A = p'(g)$, which means that A is determined up to sign. Furthermore, the relation $\epsilon'(g) = \epsilon(f(g))$ implies that

$$\rho(A^{-1}) \circ \epsilon'(g) = \epsilon([1, c]) = c \cdot I ,$$

which determines the value of c . Therefore $f(g) = [A, c]$ is uniquely determined by our conditions. □

Remark 4.2.

- (1) The triple $(G = (Spin(n) \times \mathbb{C}^\times) / K, p, \epsilon)$ is the only universal solution up to equivalence. More precisely, if (G', p', ϵ') is another universal solution, then there is a unique isomorphism $\varphi: G' \rightarrow G$ satisfying $\epsilon' = \epsilon \circ \varphi$ and $p' = p \circ \varphi$.
- (2) There is a natural Hermitian product on the representation space \mathbb{C}^k with respect to which ρ is unitary. If we require that ϵ' will be unitary, then universal solution will involve the (compact) group

$$Spin^c(n) = (Spin(n) \times U(1)) / K .$$

The universality statement and the proof are almost identical to the non-compact case.

- (3) It is important to note that although the Dirac operator is a square root of the Laplacian in the case of \mathbb{R}^n , this is no longer true in the manifold case. The Dirac operator, whose definition was outlined in §3.1, will be related to the Laplacian via the Schrödinger-Lichnerowicz formula, which involves the scalar curvature of the manifold, and the curvature of the Hermitian connection on the vector bundle S (see §3.3 in [1]).

5. SOME COROLLARIES

Denote again by $\rho: C_n \rightarrow \text{End}(\mathbb{C}^k)$ ($k = 2^{\lfloor n/2 \rfloor}$) an irreducible representation, $G = (\text{Spin}(n) \times \mathbb{C}^\times)/K$, and by $p: G \rightarrow \text{SO}(n)$ the natural homomorphism. Then Theorem 4.1 implies the following.

Corollary 5.1. *A Lie group G' and a homomorphism $p': G' \rightarrow \text{SO}(n)$ give rise to a bijection between*

$$\mathcal{A} = \left\{ \begin{array}{l} \text{Representations } \epsilon': G' \rightarrow \text{End}(\mathbb{C}^k) \text{ satisfying} \\ \epsilon'(g) \circ \rho(y) = \rho(p'(g)y) \circ \epsilon'(g) \text{ for all } g \in G', y \in \mathbb{R}^n \end{array} \right\}$$

and

$$\mathcal{B} = \{ \text{Homomorphisms } f: G' \rightarrow G \text{ such that } p' = p \circ f \}$$

Proof. Part (2) in Theorem 4.1 provides a function $f \in \mathcal{B}$ for every $\epsilon' \in \mathcal{A}$. Conversely, if $f \in \mathcal{B}$, then $\epsilon' = \epsilon \circ f$ is in \mathcal{A} , by part (1) of Theorem 4.1. \square

The above corollary provides an easy criterion for checking whether a lifting of the structure to a group G' will enable us to construct an irreducible Clifford bundle action on a vector bundle associated with this lifting. We give a few examples in the following claim.

Claim 5.1. *If $G' = U(n/2)$ (for an even n) or $G' = \text{Spin}(n)$, then there exist a homomorphism $p': G' \rightarrow \text{SO}(n)$ and a representation $\epsilon': G' \rightarrow \text{End}(\mathbb{C}^k)$ for which $\epsilon'(g) \circ \rho(y) = \rho(p'(g)y) \circ \epsilon'(g)$ for all $g \in G', y \in \mathbb{R}^n$. If $G' = \text{SO}(n)$ ($n \geq 3$) and $p': G' \rightarrow \text{SO}(n)$ is the identity, then there is no ϵ' satisfying the latter equality.*

Proof. For $G' = \text{Spin}(n)$ take p' to be the double cover of $\text{SO}(n)$, $f: \text{Spin}(n) \rightarrow G$ the inclusion, and use Corollary 5.1.

For $G' = U(n/2)$, take p' to be the standard inclusion $U(n/2) \subset \text{SO}(n)$. It is possible to define $f: U(n/2) \rightarrow G$ such that $p' = p \circ f$ (see page 27 in [1]). By the above corollary, the conclusion follows.

Finally, for $G' = \text{SO}(n)$ and $p' = \text{Id}$, if such an ϵ' would exist, the corollary implies that there is an $f: \text{SO}(n) \rightarrow G$ for which $p \circ f = \text{Id}$. This is impossible since the fundamental group of $\text{SO}(n)$ is \mathbb{Z}_2 and of G is \mathbb{Z} . \square

The above claim implies some well known facts: Every spin and every almost complex manifold is also a spin^c manifold in a natural way. Also, an irreducible Clifford module cannot be defined as a tensor bundle (i.e., as a vector bundle associated with the frame bundle of the manifold).

6. THE SYMPLECTIC CASE

For the symplectic group, a similar problem can be stated. The universal group in this case will be the complexified metaplectic group $\text{Mp}^c(n) = \text{Mp}(n) \times_{\mathbb{Z}_2} U(1)$, if we demand unitary representations, or $\text{Mp}(n) \times_{\mathbb{Z}_2} \mathbb{C}^\times$ otherwise. In this section we outline the setting in this case, and prove a similar universality statement.

6.1. Symplectic Clifford algebras. Let V be a real vector space of dimension $2n$. If $B: V \times V \rightarrow \mathbb{R}$ is a symmetric bilinear form on V , then the ideal (in the tensor algebra $T(V)$) generated by expressions of the form

$$(3) \quad v \cdot v - B(v, v) \cdot 1 \quad , \quad v \in V$$

is the same one generated by

$$(4) \quad v \cdot u + u \cdot v - 2 \cdot B(u, v) \quad , \quad u, v \in V .$$

Suppose now that $\omega: V \times V \rightarrow \mathbb{R}$ is a symplectic (i.e., an antisymmetric and non-degenerate bilinear) form on V . Since $\omega(v, v) = 0$ for all $v \in V$, we would better modify (4) and define the symplectic Clifford algebra as follows. We follow [3] and omit the coefficient ‘2’ in our definition.

Definition 6.1. The *symplectic Clifford algebra* associated with the symplectic vector space (V, ω) is defined as

$$Cl^s(V, \omega) = T(V)/I(V, \omega)$$

where $I(V, B)$ is the ideal generated by

$$\{v \cdot w - w \cdot v + \omega(v, w) \cdot 1 : v, w \in V\} .$$

Remark 6.1. If $V = \mathbb{R}^{2n}$ and ω is the standard symplectic form, given by

$$\omega(x, y) = \sum_{j=1}^n x_j y_{n+j} - x_{n+j} y_j \quad , \quad x, y \in \mathbb{R}^{2n} ,$$

then we denote

$$Cl_n^s = Cl^s(\mathbb{R}^{2n}, \omega) \quad \text{and} \quad \mathbb{C}l_n^s = Cl_n^s \otimes \mathbb{C} .$$

The symplectic Clifford algebra in this case is also called the *Weyl algebra*, and is useful since its generators satisfy relations which are similar to the relations satisfied by the position and momentum operators in quantum mechanics (see §1.4 in [3]).

Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing complex-valued smooth functions on \mathbb{R}^n . If e_1, \dots, e_{2n} is the standard basis for \mathbb{R}^{2n} , then define a linear action

$$\rho: \mathbb{R}^{2n} \rightarrow \text{End}(\mathcal{S}(\mathbb{R}^n))$$

by assigning

$$\begin{aligned} \rho(e_j)f &= i \cdot x_j f \quad \text{for} \quad j = 1, \dots, n \\ \rho(e_j)f &= \frac{\partial f}{\partial x_j} \quad \text{for} \quad j = n+1, \dots, 2n \end{aligned}$$

and extend by linearity. This action extends (see §1.4 in [3]) to a linear map

$$Cl_n^s \rightarrow \text{End}(\mathcal{S}(\mathbb{R}^n))$$

which is *not* an algebra homomorphism.

For each $v \in \mathbb{R}^{2n}$, $\rho(v)$ can be regarded as a continuous operator on the Schwartz space. We call the map ρ *Clifford multiplication*.

6.2. The metaplectic representation. The metaplectic group $Mp(n)$ will play the role that the spin group $Spin(n)$ played in the Riemannian case. The symplectic group is

$$Sp(n) = \{A \in GL(2n, \mathbb{R}) : \omega(Av, Aw) = \omega(v, w)\}$$

where ω is the standard symplectic form on \mathbb{R}^{2n} . This is a connected and non-compact Lie group.

The fundamental group of $Sp(n)$ is isomorphic to \mathbb{Z} , and thus $Sp(n)$ has a unique connected double cover, which is denoted by $Mp(n)$. Denote by

$$p: Mp(n) \rightarrow Sp(n)$$

the covering map, and by $-1 \in Mp(n)$ the nontrivial element in $\text{Ker}(p)$.

Define

$$G = (Mp(n) \times \mathbb{C}^\times) / K$$

where $K = \{(1, 1), (-1, -1)\}$. The covering map extends to a map (also denoted by p)

$$G \rightarrow Sp(n) \quad , \quad [A, z] \mapsto p(A) \quad .$$

There is an important infinite dimensional unitary representation of the metaplectic group on the Hilbert space $L^2(\mathbb{R}^n)$, which is called *the metaplectic representation*. We denote it by

$$\mathbf{m}: Mp(n) \rightarrow U(L^2(\mathbb{R}^n))$$

where $U(L^2(\mathbb{R}^n))$ is the group of unitary operators on $L^2(\mathbb{R}^n)$. For the construction of \mathbf{m} , see [3] and references therein. This representation has many interesting properties, but all we need here is the facts that the Schwartz space $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is an invariant subspace for \mathbf{m} , and is dense in $L^2(\mathbb{R}^n)$.

We extend \mathbf{m} to a representation of the group $G = (Mp(n) \times \mathbb{C}^\times) / K$, by

$$\epsilon: G \rightarrow \text{End}(L^2(\mathbb{R}^n)) \quad , \quad \epsilon([A, z]) = z \cdot \mathbf{m}(A) \quad .$$

6.3. The universality of the metaplectic group. Now we can state the universality theorem (for the group G), which turns out to be almost identical to the corresponding theorem in the Riemannian case.

Theorem 6.1. *Let $\rho: \mathbb{R}^{2n} \rightarrow \text{End}(\mathcal{S}(\mathbb{R}^n))$ be the Clifford multiplication map. Then:*

- (1) *For $G = (Mp(n) \times \mathbb{C}^\times) / K$, $p: G \rightarrow Sp(n)$ and $\epsilon: G \rightarrow \text{End}(L^2(\mathbb{R}^n))$ defined above, we have*

$$\rho(p(A)y) = \epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1}$$

for all $y \in \mathbb{R}^{2n}$ and $A \in G$ (i.e., ρ is G -equivariant).

This is an equality of operators on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

- (2) *If G' is a Lie group, $p': G' \rightarrow Sp(n)$ a group homomorphism, and $\epsilon': G' \rightarrow \text{End}(\mathcal{S}(\mathbb{R}^n))$ a continuous representation, such that*

$$\rho(p'(A)y) = \epsilon'(A) \circ \rho(y) \circ \epsilon'(A)^{-1}$$

for all $y \in \mathbb{R}^{2n}$ and $A \in G'$, then there is a unique homomorphism $f: G' \rightarrow G$ such that

$$p' = p \circ f \quad \text{and} \quad \epsilon' = \epsilon \circ f \quad .$$

Proof.

- (1) For $Mp(n)$, this is proved in Lemma 1.4.4 in [3]. The proof for G follows.
- (2) To prove the second part, we follow the same idea as in the Riemannian case. Fix an element $g \in G'$, and choose an element $A \in Mp(n)$ for which $p(A) = p'(g)$. We show that the endomorphism

$$D = \epsilon(A^{-1}) \circ \epsilon'(g) : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$$

is a nonzero complex multiple of the identity. Once this is done, the rest of the proof will be identical to the proof of Theorem 4.1, part (2).

By assumption, we have

$$\epsilon(A) \circ \rho(y) \circ \epsilon(A)^{-1} = \epsilon'(g) \circ \rho(y) \circ \epsilon'(g)^{-1}$$

which is equivalent to

$$\rho(y) \circ D = D \circ \rho(y)$$

for all $y \in \mathbb{R}^{2n}$. From the definition of ρ we conclude that D is a continuous operator on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ which commutes with all multiplication and derivative operators:

$$f \mapsto x_j \cdot f \quad \text{and} \quad f \mapsto \frac{\partial f}{\partial x_j}.$$

Such an operator must be a complex multiple of the identity. This follows from the fact that the map ρ gives rise to an irreducible representation of the symplectic Clifford algebra Cl_n^s on the space $L^2(\mathbb{R}^n; \mathbb{C})$. For a proof of this fact for $n = 1$ see Theorem 3 (page 44) in [5]. The n -dimensional case follows. □

Remark 6.2. As in the Riemannian case, if we require that ϵ' will be a unitary representation, then the group G in Theorem 6.1 will be replaced with $(Mp(n) \times \mathbb{U}(1)) / K$.

Remark 6.3. The construction of a Dirac operator in the Riemannian case was motivated by the search for a square root for the (negative) Laplacian. One may wonder what is the symplectic analog of the Dirac and the Laplacian operators. In Chapter 5 of [3] symplectic Dirac and associated second order operators are discussed. However, it is not clear to me if the *search for a square root* in the Riemannian case has a (satisfactory) symplectic analogue.

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